A fictitious domain method on Octrees/Basilisk

Can Selçuk, Anthony Wachs

University of British Columbia, Math department

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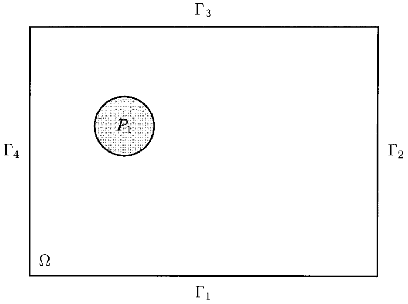
Particle laden flow with basilisk

Fictitious domain method: principle

Consider a domain Ω with boundaries $\Gamma_1, ..., \Gamma_4$ filled with (Newtonian) fluid and an (homogeneous) solid particle occupying the domain $P(t)$ with boundary $\partial P(t)$:



fluid domain: $\Omega \backslash \overline{P(t)}$

solid domain: $P(t)$ with boundary $\partial P(t)$

Reference: (Glowinski et al., 1999).

Fictitious domain method: starting point (in strong form)

- combined-equations of motion with Lagrange multipliers $\boldsymbol{\lambda}$:

$$\rho_L \left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\, \boldsymbol{u} \right) = -\boldsymbol{\nabla} p + \boldsymbol{\nabla} \cdot (2\mu \boldsymbol{D}) - \boldsymbol{\lambda} \text{ in } \Omega,$$

$$\left(1 - \frac{\rho_L}{\rho_s}\right) \left(M \left(\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t} - \boldsymbol{g} \right) \right) = \int_{P(t)} \boldsymbol{\lambda}\, \mathrm{d}\boldsymbol{x} \text{ in } P(t),$$

$$\left(1 - \frac{\rho_L}{\rho_s}\right) \left(\boldsymbol{I} \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} + \boldsymbol{\omega} \times \boldsymbol{I}\boldsymbol{\omega} \right) = \int_{P(t)} \boldsymbol{r} \times \boldsymbol{\lambda}\, \mathrm{d}\boldsymbol{x} \text{ in } P(t)$$

$$\boldsymbol{u} - (\boldsymbol{U} + \boldsymbol{\omega} \times \boldsymbol{r}) = 0 \text{ over } P(t)$$

- continuity equation

$$-\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \text{ over } \Omega$$

- unknowns: $\boldsymbol{u}, p, \boldsymbol{U}, \boldsymbol{\omega}, \boldsymbol{\lambda}$

Operator splitting

The process can be seen as a particular initial value problem:

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} + \mathbf{NS}\left(\phi\right) + \mathbf{Gra}\left(\phi\right) + \mathbf{Fd}\left(\phi\right) = f$$
$$\phi(t=0) = \phi_0.$$

Split in 3 and solve successively (Glowinski et al., 1999):

$$\frac{\phi^{n+1/3} - \phi^{n}}{\Delta t} + \mathbf{NS}\left(\phi^{n+1/3}\right) = f_1^{n+1},$$
$$\frac{\phi^{n+2/3} - \phi^{n+1/3}}{\Delta t} + \mathbf{Gra}\left(\phi^{n+2/3}\right) = f_2^{n+1},$$
$$\frac{\phi^{n+1} - \phi^{n+2/3}}{\Delta t} + \mathbf{Fd}\left(\phi^{n+1}\right) = f_3^{n+1},$$

with $f_1^{n+1} + f_2^{n+1} + f_3^{n+1} = f\left((n+1)\Delta t\right).$

Operator splitting

Pros:

- Flexible for the choice of each sub-problem's solvers: use any available solver in your group/internet.
- Relatively easy to implement
- Robust (stable) and preserves stationnary solutions (MacNamara and Strang, 2016)

Cons:

- First order accurate only

Validation: Stokes flow through a periodic array of spheres

Tri-periodic domain. Flow initially at rest, motion imposed with a pressure gradient.

Drag coefficint K, D diameter, ϕ concentration, V superficial velocity

$$F_i = 3\pi\mu DKV_i$$

$$D/2 = (3\phi/4\pi)^{1/3}$$

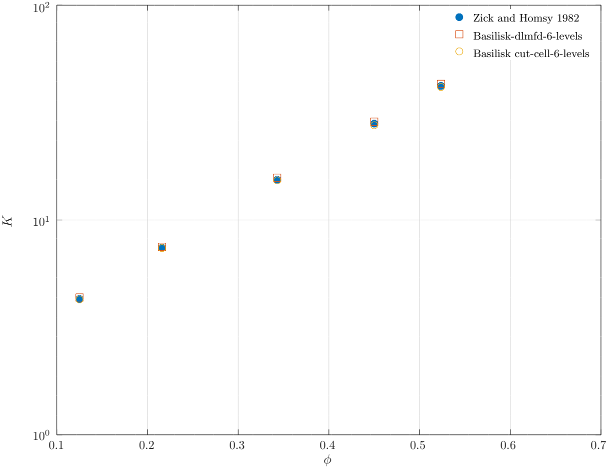
$$V_i = \frac{1}{\tau_0}\iiint_{\Omega/P} u_i(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$$

Zick and Homsy (1982)

streamwise velocity

Validation: Stokes flow through a periodic array of spheres

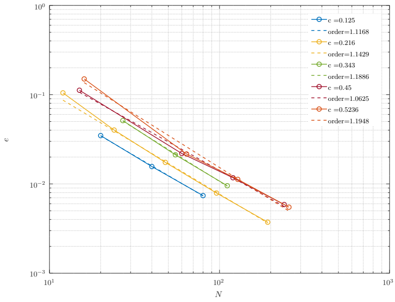
Validation of octrees with constant 2^6 cells per direction.



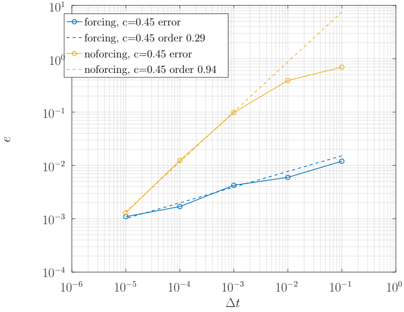
Drag coefficient K as a function of the concentration ϕ.

Validation: Stokes flow through a periodic array of spheres

First order convergence rate in space and time.



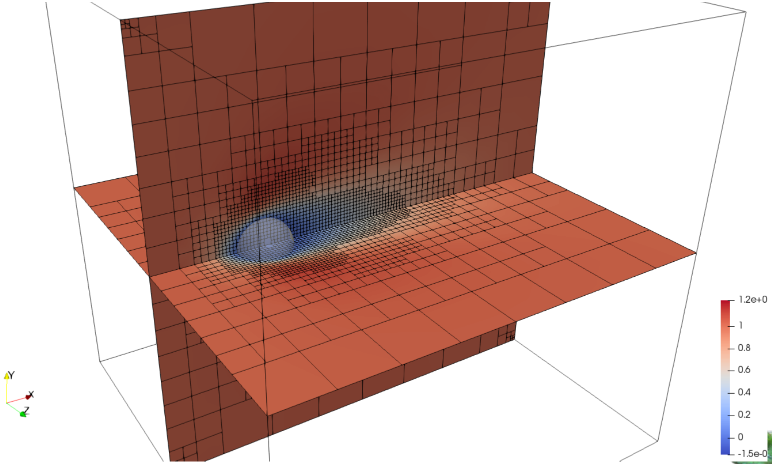
Spatial convergence



Temporal convergence

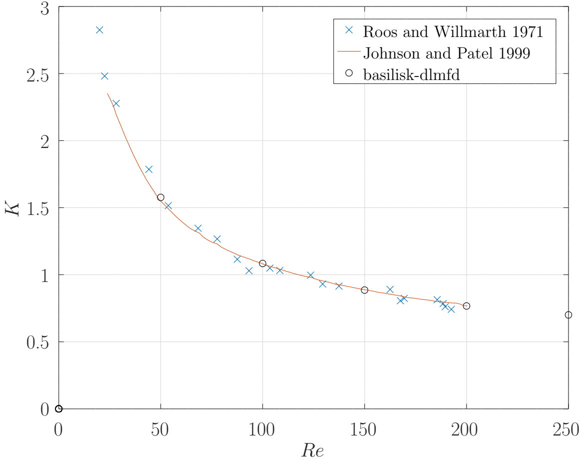
Validation: Flow past a sphere, adaptive meshes

Flow past a sphere at $50 \leq Re \leq 250$ with $N = 9, \ldots, 13$ level of refinement ($34 - 136$ points per diameter). Box size $L/D = 30$.



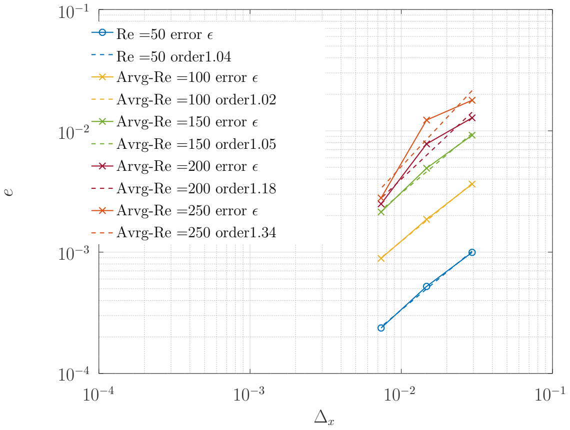
Validation: Flow past a sphere, adaptive meshes

Comparison of the drag coefficient $K(Re)$ against previous works.



Validation: Flow past a sphere, adaptive meshes

Spatial convergence rate: ranging from 1.04 to 1.34.



Validation: flow past a cylinder at Re $= 9500$

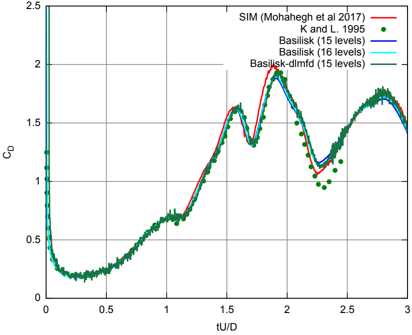
15 level of refinement, domain size $L/D = 18$, resolution $2^{15}/18 \sim 1820 pts/D$. Equivalent cartesian grid $2^{30} \sim 10^9$cells

Axial vorticity $\omega_z\,(t)$

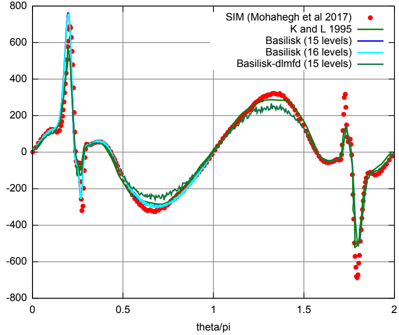
Animation of the mesh

Validation: flow past a cylinder at Re = 9500

Comparison with respect to other codes/papers/techniques:



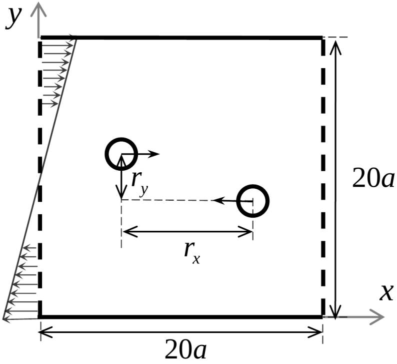
Drag coefficient C_D



Surface vorticty at $t = 2.5$

More at http://basilisk.fr/sandbox/cselcuk/starting-dlmfd.c

Two moving spheres in creeping shear flow



$\vec{r}$: inter-particle distance

a: sphere's radius

"Analytical" solution:

$$\frac{\mathrm{d}r_x}{\mathrm{d}t} = r_y + er_x - \frac{B(\vec{r})}{2} r_y,$$
$$\frac{\mathrm{d}r_y}{\mathrm{d}t} = er_y - \frac{B(\vec{r})}{2} r_x,$$
$$\frac{\mathrm{d}r_z}{\mathrm{d}t} = er_z,$$

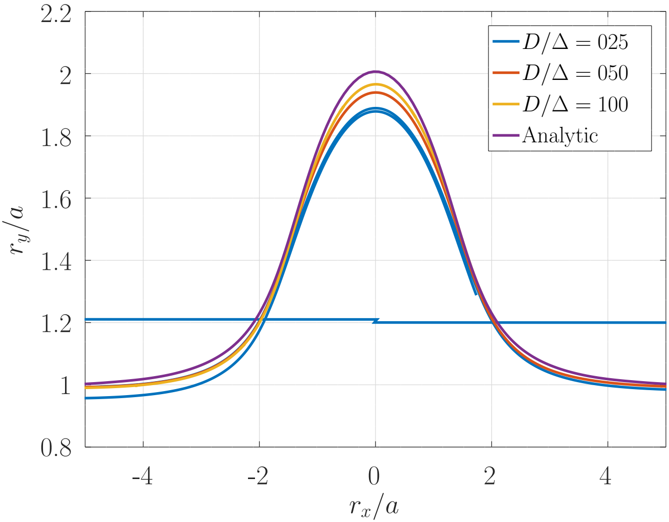
where

$$e = \frac{r_x r_y \left(B(\vec{r}) - A(\vec{r})\right)}{\vec{r}^2},$$

(Batchelor and Green, 1972) and (Lin et al., 1970)

Two moving spheres in creeping shear flow

Time-step $\Delta t = 1/3000$:



Two moving cylinders in creeping shear flow

2D: Time-step $\Delta t = 1/3000$, mesh size $D/\Delta \sim 200$, 11 levels of refinnement

refinement criterion on $\vec{u}$ and the color field

refinement criterion only on the color field

Attempt to capture lubrication forces: preliminary conclusion

Globally encouraging results:

- Lubrication force almost fully captured by brute-forcing
- No contact and sub-grid models
- Robust even when particles overlap

Dynamic of multiple particles with Grains 3D

Coupling with Grains3D as granular solver (C++ code) with Basilisk's Navier-Stokes solver (C code).

Dynamic of a free falling cube, $\rho_s/\rho_s = 7$, $Ga = 140$

Box size $L/D = 700$, (equivalent) spatial resolution $\sim 25 pts/D$.

Dynamic of cluster of (600) particles at $Re = 15$

Simulations performed by Daniel Oliveira (L.S.U)

Basilisk dlmfd + grain3D

Experiments by (Pignatel et al., 2011)

Thank you

Thank you for your attention !

More on my sandbox at: http://basilisk.fr/sandbox/cselcuk/

The first, Navier-Stokes problem

Basilisk's global temporal-scheme for the Navier-Stokes problem reads: given $\boldsymbol{u}^n, \boldsymbol{\lambda}^n$, find $\boldsymbol{u}^{n+1/2}, p^{n+1/2}$ such that:

$$\frac{\boldsymbol{u}^{n+1/2} - \boldsymbol{u}^n}{\Delta t} = -\left[\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}\right]^{n+1/4} + \frac{1}{\rho_f}\left[\boldsymbol{\nabla} \cdot \left(2\mu \boldsymbol{\mathcal{D}}_{\boldsymbol{v}}^{n+1/2}\left[\boldsymbol{u}^{n+1/2}\right]\right) - \boldsymbol{\nabla} p^{n+1/2}\right] - \boldsymbol{\lambda}^n,$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}^{n+1/2} = \boldsymbol{0}.$$

Solved with a modified version of the projection scheme proposed by (Bell et al., 1989).

The second, granular problem

The second sub-problem is a pure granular problem which reads: given $\boldsymbol{U_i^n}, \boldsymbol{\omega_i^n}$ find $\boldsymbol{U_i^{n+1/2}}, \boldsymbol{\omega_i^{n+1/2}}$ such that

$$\left(1 - \frac{\rho_f}{\rho_s}\right) M \left(\frac{\boldsymbol{U}^{n+1/2} - \boldsymbol{U}^n}{\Delta t}\right) = \left(1 - \frac{\rho_f}{\rho_s}\right) M\boldsymbol{g} + \boldsymbol{Fc}$$

$$\left(1 - \frac{\rho_f}{\rho_s}\right) \boldsymbol{I} \left(\frac{\boldsymbol{\omega}^{n+1/2} - \boldsymbol{\omega}^n}{\Delta t}\right) = -\left(1 - \frac{\rho_f}{\rho_s}\right) \boldsymbol{\omega}^n \times \boldsymbol{I\omega}^n + \boldsymbol{Tc}.$$

Can be solved with any granular solver that handles contact forces and torques for multiple particles.

The third, fictitious-domain problem

The fictitious-domain problem reads: given $\boldsymbol{u}^{n+1/2}$, $\boldsymbol{\lambda}^{n}$, $\boldsymbol{U}^{n+1/2}$, $\boldsymbol{\omega}^{n+1/2}$ solve

$$\rho_f \left(\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^{n+1/2}}{\Delta t} \right) - \boldsymbol{\lambda}^{n+1} = \boldsymbol{\lambda}^{n} \text{ over } \Omega$$

$$\left(1 - \frac{\rho_f}{\rho_s}\right) \left(M \left[\frac{\boldsymbol{U}^{n+1} - \boldsymbol{U}^{n+1/2}}{\Delta t} \right] \right) = - \int_{P(t)} \boldsymbol{\lambda}^{n+1} \, \mathrm{d}\boldsymbol{x} \text{ over } P(t)$$

$$\left(1 - \frac{\rho_f}{\rho_s}\right) \left(\boldsymbol{I} \left(\frac{\boldsymbol{\omega}^{n+1} - \boldsymbol{\omega}^{n+1/2}}{\Delta t} \right) \right) = - \int_{P(t)} \boldsymbol{r} \times \boldsymbol{\lambda}^{n+1} \, \mathrm{d}\boldsymbol{x} \text{ over } P(t)$$

$$\boldsymbol{u}^{n+1} - \left(\boldsymbol{U}^{n+1} + \boldsymbol{\omega}^{n+1} \times \boldsymbol{r} \right) = \boldsymbol{0} \text{ over } P(t).$$

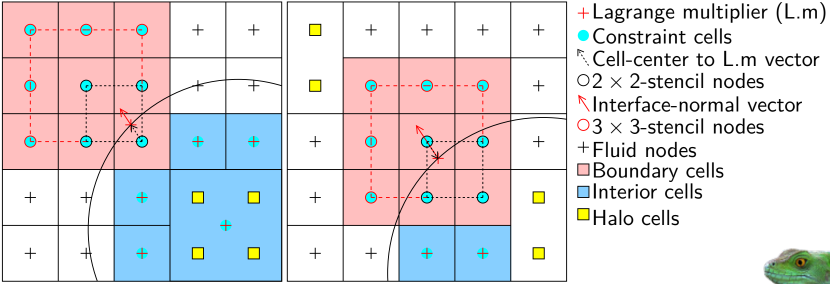
Saddle-point problem solved with an iterative algorithm (Uwaza).

Interface reconstruction

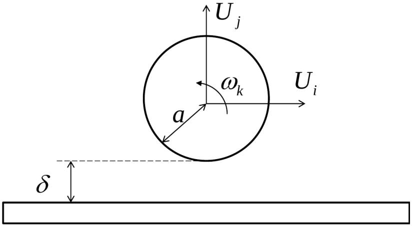
Collocation point method: use of a Dirac delta function as basis function for the Lagrange multipliers $\boldsymbol{\lambda}$:

$$\boldsymbol{\lambda}(\boldsymbol{x}) = \sum_{l=1}^{L} \boldsymbol{\lambda_l} \delta(\boldsymbol{x} - \boldsymbol{x_l}) . \tag{1}$$

with L the number of Lagrange multipliers.



One sphere close to wall in Stokes flow



δ: gap distance
a: sphere's radius

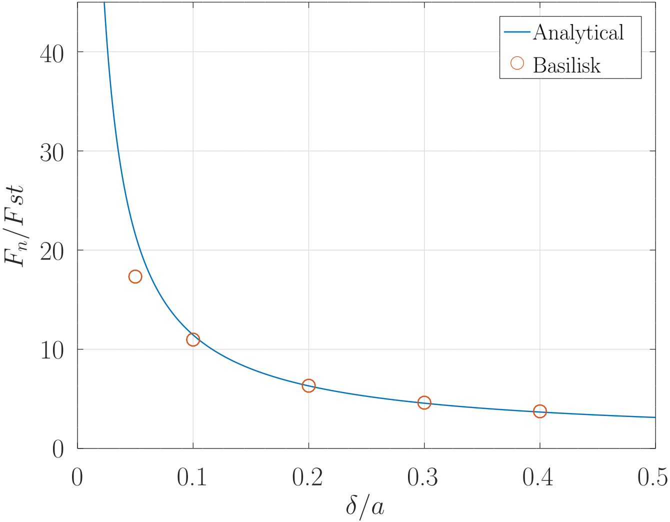
- Large box: $L/a = 60$
- Periodicity on front/back and left/right faces
- "Wall" for bottom/top faces
- fixed particle
- imposed velocity: $\vec{U} = (0, -U_c, 0)$, $\vec{\omega} = \vec{0}$
- $T_c = 2a/U_c$

Analytical solution:

$$F_n/F_{st} = (\delta/a)^{-1} - \frac{1}{5}\log(\delta/a) + 0.97128, \qquad (2)$$

(Brenner, 1961) and (Cooley and O'Neill, 1969)

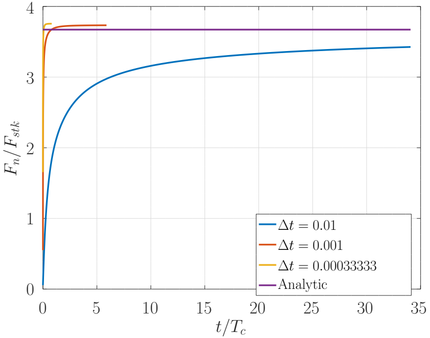
One sphere close to wall in Stokes flow



One sphere close to wall: temporal convergence

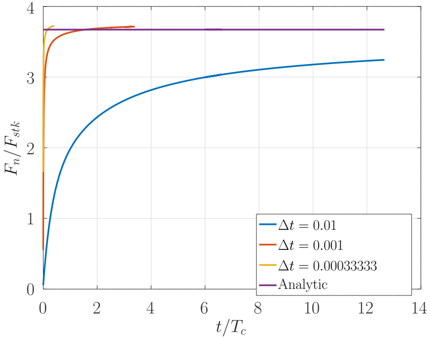
case: $\delta/a = 0.4$:

$a/\Delta \sim 34$, $\delta/\Delta \sim 25$



11 levels of refinement

$a/\Delta \sim 68$, $\delta/\Delta \sim 50$

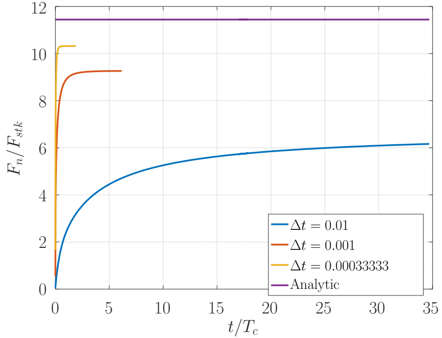


12 levels of refinement

One sphere close to wall: temporal convergence

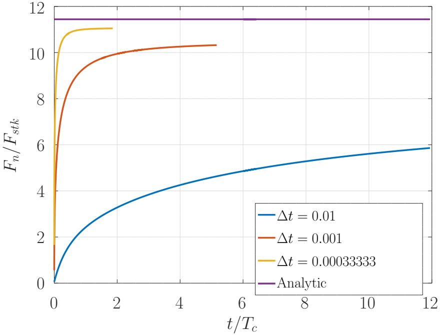
More challenging case: $\delta/a = 0.1$:

$a/\Delta \sim 34$, $\delta/\Delta \sim 3$



11 levels of refinement

$a/\Delta \sim 68$, $\delta/\Delta \sim 6$



12 levels of refinement